SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2006
TRIAL HIGHER SCHOOL CERTIFICATE

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes.
- Working time – 3 hours.
- Write using black or blue pen.
- Board approved calculators may be used.
- All *necessary* working should be shown in every question if full marks are to be awarded.
- Marks may **NOT** be awarded for messy or badly arranged work.
- Hand in your answer booklets in **3** sections.
  Section **A** (Questions 1 - 2),
  Section **B** (Questions 3 - 4)
  Section **C** (Questions 5 - 6)
  Section **D** (Questions 7 –8).
- Start each **NEW** section in a separate answer booklet.

Total Marks - 120 Marks

- Attempt Sections A - D
- All questions are of equal value.

Examiner: *E. Choy*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

Total marks – 120
Attempt Questions 1 - 8
All questions are of equal value

Answer each section in a SEPARATE writing booklet. Extra writing booklets are available.

SECTION A (Use a SEPARATE writing booklet)

Question 1 (15 marks) **Marks**

(a) By first completing the square, evaluate the following integrals

(i) $\displaystyle\int_{-1}^{0} \frac{dx}{\sqrt{3-2x-x^2}}$ **2**

(ii) $\displaystyle\int_{0}^{1} \sqrt{x(1-x)}\,dx$ **2**

(b) Integrate the expressions below

(i) $\displaystyle\int \frac{1}{x\ln x}\,dx$ **1**

(ii) $\displaystyle\int x\ln x\,dx$ **2**

(iii) $\displaystyle\int \frac{x+1}{x^2+x+1}\,dx$ **2**

(c) Use the technique of *integration by parts* to evaluate **2**

$$\int_{0}^{\frac{1}{2}} \cos^{-1} x\,dx$$

(d) (i) Find real numbers A, B, and C so that **2**

$$\frac{10}{(3+x)(1+x^2)} = \frac{A}{3+x} + \frac{Bx+C}{1+x^2}$$

for all $x \neq -3$

(ii) Use part (i) above and the substitution $t = \tan\theta$ to find **2**

$$\int \frac{10\,d\theta}{3+\tan\theta}$$

SECTION A continued

Question 2 (15 marks) **Marks**

(a) (i) Write the complex number $-\sqrt{3}+i$ in modulus-argument form. 1

(ii) Hence, use de Moivre's Theorem to find $\left(-\sqrt{3}+i\right)^{10}$ in the form $a+ib$, for real values a and b. 2

(b) Sketch each of the following regions on separate Argand diagrams

(i) $-1<\text{Re}(z)<2$ and $0<\text{Im}(z)<3$ 2

(ii) $z\overline{z}-(1-i)z-(1+i)\overline{z}<2$ 2

(iii) $0<\arg\left[(1-i)z\right]<\frac{\pi}{6}$ 2

(c) (i) Find the square roots of the complex number $-3+4i$ 2

(ii) Find the roots of the quadratic equation $x^2-(4-2i)x+(6-8i)=0$ 2

(d) The locus of a point P, which moves in the complex plane, is represented by the equation $|z-(3+4i)|=5$

(i) Sketch the locus of the point P. 2

(ii) Find the modulus of z when $\arg z=\tan^{-1}\left(\frac{1}{2}\right)$. 2

SECTION B (Use a SEPARATE writing booklet)

Question 3 (15 marks)

(a) Find a cubic equation with roots α, β and γ such that 3

$$\left.\begin{aligned}\alpha\beta\gamma&=5\\ \alpha+\beta+\gamma&=7\\ \alpha^2+\beta^2+\gamma^2&=29\end{aligned}\right\}$$

(b) The polynomial $P(x)$ is defined by $P(x)=x^4+Ax^2+B$, where A and B are real positive numbers.

(i) Explain why $P(x)$ has no real zeroes. 2

(ii) If two of the zeroes of $P(x)$ are ib and $-id$ where b and d are real show that: 4

$$b^4+d^4=A^2-2B$$

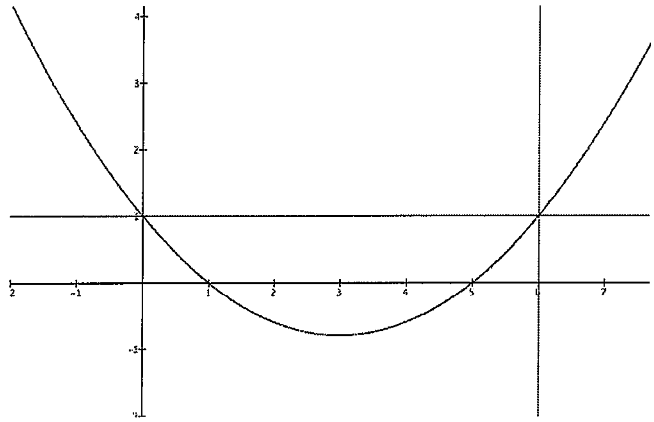
(c) Given that $f(x)=x^3-3ax+b$, where a and b are real numbers then:

(i) Show that $y=f(x)$ has turning points if $a>0$, and find their coordinates. 3

(ii) Show that $f(x)$ has three distinct real zeroes if $b^2<4a^3$. 3

Question 4 (15 marks) **Marks**

(a)



The sketch above shows the parabola $y = f(x)$, where

$$f(x) = \frac{1}{5}(x-1)(x-5).$$

Without any use of calculus, draw careful sketches of the following curves, showing all intercepts, asymptotes and turning points.

NB The vertex of the parabola is at $\left(3, -\frac{4}{5}\right)$.

(i) $y = \dfrac{1}{f(x)}$ **2**

(ii) $y = [f(x)]^2$ **3**

(iii) $y = \tan^{-1}[f(x)]$ **3**

(iv) $y = f(\ln x)$ **3**

(b) Suppose the function $f(x) = O(x) + E(x)$, where $O(x)$ is odd and $E(x)$ is even.

(i) By considering $f(-x)$, find an expression for $O(x)$ in terms of f. **2**

(ii) Hence write down $O(x)$ when $f(x) = e^x$. **2**

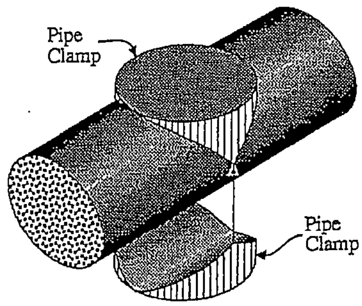
SECTION C starts on page 5

SECTION C (Use a SEPARATE writing booklet)

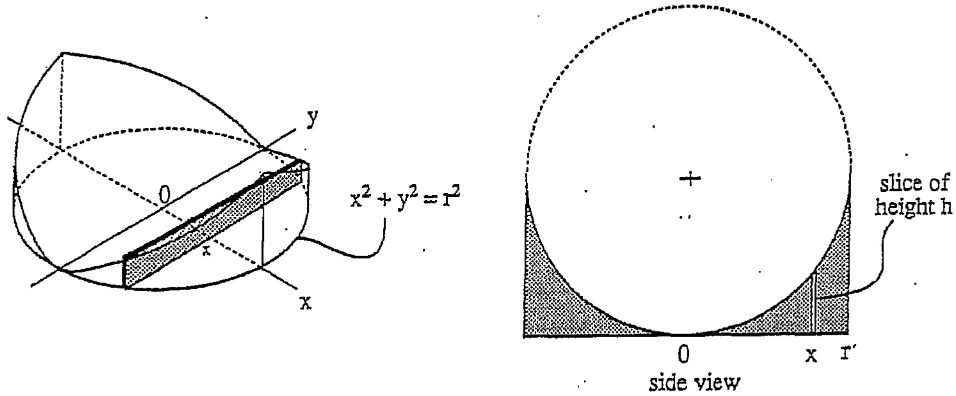
Question 5 (15 marks) **Marks**

(a) A pipe-clamp is made of two identical pieces. Each piece has a circular base of radius r units and the other face is curved so as to fit flush against the pipe held between the two pieces.

The pipe also has a radius of r units.



A vertical slice, of thickness Δx, taken x units from the centre of the base is in the shape of a rectangle with one side in the circular base and of height necessary to reach the cylindrical pipe as shown in the diagram below:



(i) Show that the height of the slice taken x units from O is given by **3**

$$h = r - \sqrt{r^2 - x^2}$$

(ii) Show that the volume, ΔV, of such a slice is given by **3**

$$\Delta V \approx \left[2r\sqrt{r^2 - x^2} - 2\left(r^2 - x^2\right)\right]\Delta x$$

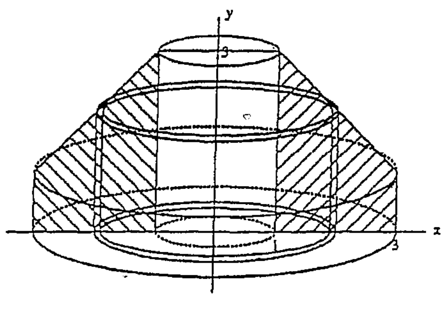
(iii) Hence, find by integration, the volume of ONE piece of the pipe-clamp. **3**

(b) (i) Show that the volume, ΔV, of a right cylindrical shell of height H, with inner radius r and thickness Δr is given by the formula **2**

$$\Delta V = 2\pi r H \Delta r$$

where Δr is sufficiently small so that $(\Delta r)^2$ may be neglected.

(b) (ii) A metal umbrella base is formed by rotating the area enclosed between $x = 1$, $x = 3$, $y = 0$ and $y = 4 - x$ about the y-axis as shown. **4**



Using the method of cylindrical shells, find the volume of the umbrella base.

Question 6 (15 marks)

(a) A point T moves so that the sum of its distances from the point $(-2,0)$ and $(2,0)$, on a Cartesian plane, is 6 units.

(i) Show that the locus of T is an ellipse $\mathscr{E}$ with the equation **2**

$$\frac{x^2}{9}+\frac{y^2}{5}=1$$

(ii) Find the equation of the auxiliary circle, $\mathscr{A}$, of $\mathscr{E}$. **1**

(iii) Find the eccentricity, coordinates of the foci and the equations of the directrices of the ellipse, $\mathscr{E}$. **2**

(iv) Draw a neat sketch, showing the ellipse and its auxiliary circle. **1**

(v) A line parallel to the y-axis meets the positive x-axis at N and the curves $\mathscr{E}$ and $\mathscr{A}$ at P and Q respectively. **1**
Given the coordinates $N(3\cos\theta,0)$, find the coordinates of P and Q (where P and Q are in the first quadrant).

(vi) Find the equations of the tangents at P and Q. **2**

(vii) If R is the point of intersection of the tangents at P and Q:

(α) Show that R lies on the major axis of $\mathscr{E}$. **2**

(β) Prove that the product of the lengths ON and OR is independent of the positions of P and Q on the curves. **2**

(b) Given p red balls and m yellow balls, where $p-m+1>0$, arranged in a row. Show that the number of ways of arranging them so that no two yellow balls appear together is given by: **2**

$$^{p+1}C_m$$

SECTION D starts on page 7

SECTION D (Use a SEPARATE writing booklet)

Question 7 (15 marks) **Marks**

(a) (i) Show that $z^5+1=(z+1)(z^4-z^3+z^2-z+1)$ 1

(ii) If z is a solution to $z^5+1=0$ where $z\neq -1$, prove that $1+z^2+z^4=z+z^3$. 1

(iii) Hence show that $\cos\frac{\pi}{5}+\cos\frac{3\pi}{5}=\frac{1}{2}$ 3

(b) For integer values of k where $k=0,1,2,\ldots$ define I_k as follows:

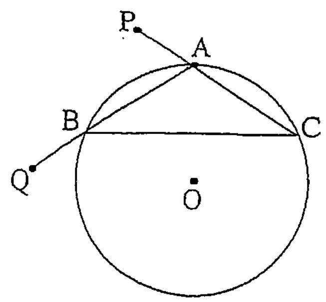
$$I_k=\int_0^{\frac{\pi}{2}}\cos^k x\,dx$$

(i) Express I_{k+2} in terms of k and I_k. 2

(ii) Hence find an expression for I_{2n}, where $n=0,1,2,\ldots$ 2

(c) In ΔABC, in the diagram on the right, $AB=AC$.

Produce CA to P and AB to Q so that $AP=BQ$.



(i) Show that $\angle OAP=\angle OBQ$. 3

(ii) Prove that A, P, Q and O, the centre of circle ABC, are concyclic. 3

Question 8 (15 marks)

(a) A particle is projected **vertically** upwards in a resisting medium where the resistance varies as the square of the velocity and k is the constant of variation. If the velocity of projection is $v_0\tan\alpha$,

(i) Show that the maximum height, H, reached is given by: 3

$$H=\frac{1}{2k}\ln\left(\frac{g+kv_0^2\tan^2\alpha}{g}\right)$$

(ii) Show that the particle returns to the point of projection with velocity $v_0\sin\alpha$ given that v_0 is the terminal velocity. 4

(iii) Show that the time of ascent is $\dfrac{v_0\alpha}{g}$ 3

(iv) Show that the time of descent is $\dfrac{v_0}{g}\ln(\sec\alpha+\tan\alpha)$ 2

(b) Prove by induction that, for all integers n where $n>1$, that 3

$$\frac{4^n}{n+1}<\frac{(2n)!}{(n!)^2}$$

End of paper

1) a) i) $\displaystyle\int_{-1}^{0} \frac{dx}{\sqrt{3-2x-x^2}}$

$= \displaystyle\int_{-1}^{0} \frac{dx}{\sqrt{-(x^2+2x+1)+4}}$

$= \displaystyle\int_{-1}^{0} \frac{dx}{\sqrt{4-(x+1)^2}}$

$= \left[\sin^{-1}\left(\frac{x+1}{2}\right)\right]_{-1}^{0}$

$= \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(0)$

$= \frac{\pi}{6}$

ii) $\displaystyle\int_{0}^{1} \sqrt{x-x^2}\,dx$

$= \displaystyle\int_{0}^{1} \sqrt{-(x^2-x+\tfrac{1}{4})+\tfrac{1}{4}}\,.\,dx$

$= \displaystyle\int_{0}^{1} \sqrt{\tfrac{1}{4}-(x-\tfrac{1}{2})^2}\;dx$

let $x - \frac{1}{2} = \frac{1}{2}\sin\theta$

$\frac{dx}{d\theta} = \frac{1}{2}\cos\theta$

$dx = \frac{1}{2}\cos\theta\,d\theta$

when $x=1$, $\theta = \frac{\pi}{2}$

$x=0$, $\theta = -\frac{\pi}{2}$

$= \displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\tfrac{1}{4}-\tfrac{1}{4}\sin^2\theta}\,.\,\tfrac{1}{2}\cos\theta\,d\theta$

$= \frac{1}{4}\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1-\sin^2\theta}\,.\,\cos\theta\,d\theta$

$= \frac{1}{4}\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\,d\theta$

(since even)

$= \frac{1}{2}\displaystyle\int_{0}^{\frac{\pi}{2}} \cos^2\theta\,d\theta$

$= \frac{1}{4}\displaystyle\int_{0}^{\frac{\pi}{2}} (1+\cos 2\theta)\,d\theta$

$= \frac{1}{4}\left[\theta + \frac{1}{2}\sin 2\theta\right]_{0}^{\frac{\pi}{2}}$

$= \frac{1}{4}\left[\frac{\pi}{2}\right]$

$= \frac{\pi}{8}$

OR

let $u = x - \frac{1}{2}$

$\frac{du}{dx} = 1$

$dx = du$

when $x=1$, $u = \frac{1}{2}$

$x=0$, $u = -\frac{1}{2}$

$= \displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{\tfrac{1}{4}-u^2}\;du$

Area of semicircle radius $\frac{1}{2}$.

$= \frac{1}{2}\pi\left(\frac{1}{2}\right)^2$

$= \frac{\pi}{8}$

b) i) $\displaystyle\int \frac{1}{x \ln x}\, dx$

$$= \int \frac{\left(\frac{1}{x}\right)}{\ln x}\, dx$$

$$= \ln(\ln x) + C$$

ii) $\displaystyle\int x \ln x \, dx$

$u = \ln x \qquad v' = x$

$u' = \frac{1}{x} \qquad v = \frac{x^2}{2}$

$$= \frac{x^2}{2}\ln x - \int \frac{x}{2}\, dx$$

$$= \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$$

iii) $\displaystyle\int \frac{x+1}{x^2+x+1}\, dx$

$$= \frac{1}{2}\int \frac{2x+1}{x^2+x+1}\, dx + \frac{1}{2}\int \frac{dx}{x^2+x+1}$$

$$= \frac{1}{2}\ln(x^2+x+1) + \frac{1}{2}\int \frac{dx}{\left(x^2+x+\frac{1}{4}\right)+\frac{3}{4}}$$

$$= \frac{1}{2}\ln(x^2+x+1) + \frac{1}{2}\int \frac{dx}{\frac{3}{4}+\left(x+\frac{1}{2}\right)^2}$$

$$= \frac{1}{2}\ln(x^2+x+1) + \frac{1}{2}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C$$

$$= \frac{1}{2}\ln(x^2+x+1) + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

c) $\int_0^{\frac{1}{2}} 1.\cos^{-1}x\,dx$

$u=\cos^{-1}x \qquad v'=1$

$u'=\dfrac{-1}{\sqrt{1-x^2}} \qquad v=x$

$= \left[x\cos^{-1}x\right]_0^{\frac{1}{2}} + \int_0^{\frac{1}{2}} \dfrac{x}{\sqrt{1-x^2}}$

let $u=1-x^2$

$\dfrac{du}{dx}=-2x$

$dx=\dfrac{du}{-2x}$

when $x=\frac{1}{2}$, $u=\frac{3}{4}$

$x=0$, $u=1$

$= \frac{1}{2}\cos^{-1}\left(\frac{1}{2}\right) - 0 + \int_1^{3/4} \dfrac{x}{\sqrt{u}} \cdot \dfrac{du}{-2x}$

$= \frac{1}{2}\cdot\frac{\pi}{3} - \frac{1}{2}\int_1^{3/4} u^{-\frac{1}{2}}\,du$

$= \dfrac{\pi}{6} - \dfrac{1}{2}\left[2u^{\frac{1}{2}}\right]_1^{3/4}$

$= \dfrac{\pi}{6} - \dfrac{1}{2}\left[2.\dfrac{\sqrt{3}}{2} - 2\right]$

$= \dfrac{\pi}{6} + \dfrac{2-\sqrt{3}}{2}$

d) i) $\dfrac{10}{(3+x)(1+x^2)} = \dfrac{A}{3+x} + \dfrac{Bx+C}{1+x^2}$

$10 = A(1+x^2) + (Bx+C)(3+x)$

when $x=-3$

$10=10A$

$A=1$

equate coefficients of x^2

$0=1+B$

$B=-1$

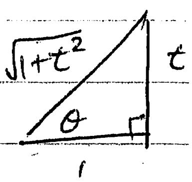
equate constants

$10=1+3C$

$3C=9$

$C=3$

ii) $\int \dfrac{10\,d\theta}{3+\tan\theta}$



$t=\tan\theta$

$\dfrac{dt}{d\theta}=\sec^2\theta$

$d\theta=\dfrac{dt}{\sec^2\theta}$

$d\theta=\cos^2\theta\,dt$

$d\theta=\dfrac{dt}{1+t^2}$

$= \int \dfrac{10}{3+t}\cdot\dfrac{dt}{1+t^2}$ using(i)

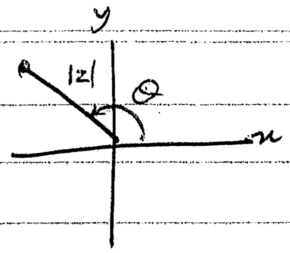
$= \int \dfrac{dt}{3+t} + \int \dfrac{-t+3}{1+t^2}\,dt$

$= \int \dfrac{dt}{3+t} - \frac{1}{2}\int \dfrac{2t}{1+t^2}\,dt + 3\int \dfrac{dt}{1+t^2}$

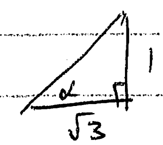
$= \ln(3+t) - \frac{1}{2}\ln(1+t^2) + 3\tan^{-1}t + C$

$= \ln(3+\tan\theta) - \frac{1}{2}\ln(1+\tan^2\theta) + 3\theta + C$

2) a) i)



$$|z| = \sqrt{(-\sqrt{3})^2 + (1)^2}$$
$$= \sqrt{4}$$
$$= 2$$



$\tan\alpha = \frac{1}{\sqrt{3}}$

$\alpha = \frac{\pi}{6}$

$\theta = \frac{5\pi}{6}$

$\therefore -\sqrt{3} + i = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

ii) $(-\sqrt{3}+i)^{10} = \left[2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)\right]^{10}$

$$= 2^{10}\left(\cos\frac{50\pi}{6} + i\sin\frac{50\pi}{6}\right)$$

$$= 1024\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$= 1024\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$

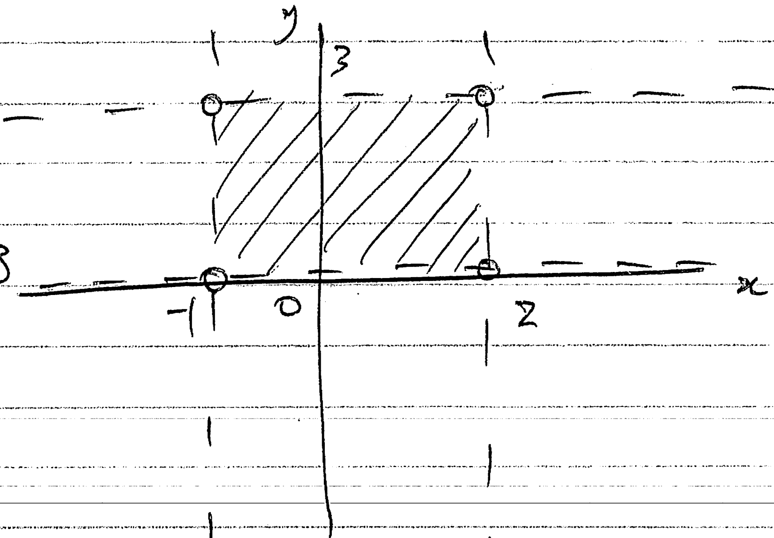
$$= 512 + 512\sqrt{3}i$$

b) i) let $z = x+iy$

$-1 < Re(z) < 2$, $0 < Im(z) < 3$

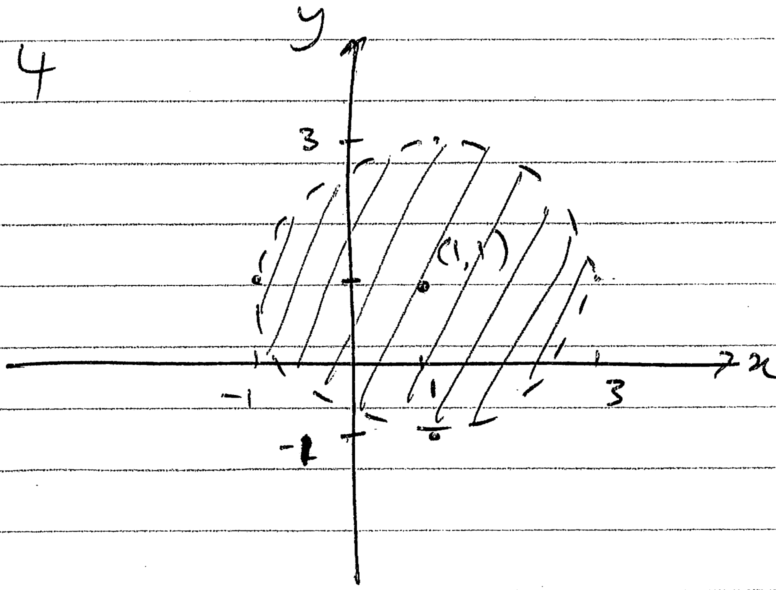
$-1 < Re(x+iy) < 2$, $0 < Im(x+iy) < 3$

$\therefore -1 < x < 2$, $0 < y < 3$



ii) let $z = x+iy$

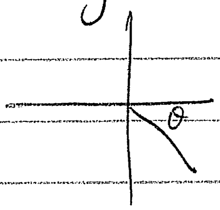
$$z\bar{z} - (1-i)z - (1+i)\bar{z} < 2$$
$$(x+iy)(x-iy) - (1-i)(x+iy) - (1+i)(x-iy) < 2$$
$$x^2+y^2 - (x+iy-ix+y) - (x-iy+ix+y) < 2$$
$$x^2+y^2 - x - \cancel{iy} + \cancel{ix} - y - x + \cancel{iy} - \cancel{ix} - y < 2$$
$$x^2 - 2x + 1 + y^2 - 2y + 1 < 2+1+1$$
$$(x-1)^2 + (y-1)^2 < 4$$



iii) $0 < \arg\left[(1-i)z\right] < \frac{\pi}{6}$

* When two complex numbers are multiplied together the argument of the resulting complex number is the sum of the individual arguments

$$\arg(1-i) = -\frac{\pi}{4}$$



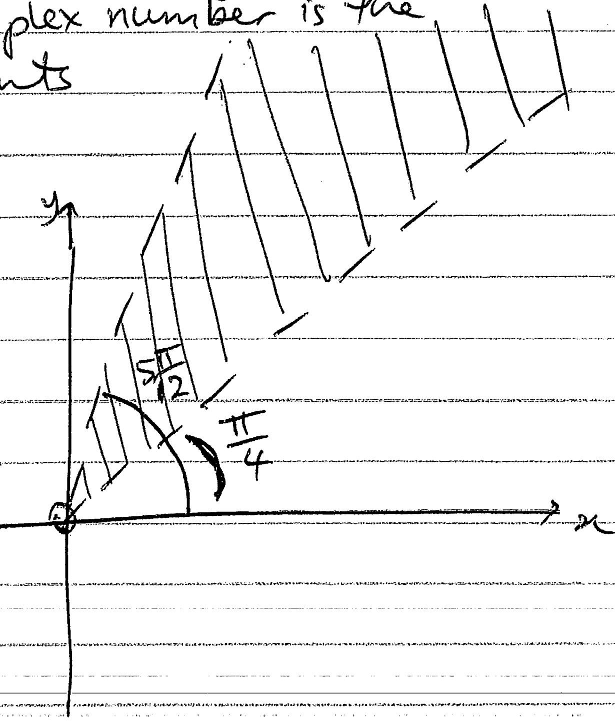
let $z = r(\cos\theta + i\sin\theta)$

$$\arg(z) = \theta$$

$$0 < \arg\left[(1-i)z\right] < \frac{\pi}{6}$$

$$0 < \theta - \frac{\pi}{4} < \frac{\pi}{6}$$

$$\frac{\pi}{4} < \theta < \frac{5\pi}{12}$$



c) i) let $\sqrt{-3+4i} = x+iy$ where x & y are real.

$$-3+4i = x^2-y^2+2xyi$$

equating real & imaginary parts

$x^2-y^2=-3$ ——— (1)

$2xy = 4$ ——— (2)

rearrange (2) $y = \frac{2}{x}$ — (2a)

sub (2a) into (1)

$$x^2-\left(\frac{2}{x}\right)^2=-3$$

$$x^2-\frac{4}{x^2}=-3$$

$$x^4-4=-3x^2$$

$$x^4+3x^2-4=0$$

$$(x^2+4)(x^2-1)=0$$

$x^2=-4$ $\quad$ $x^2=1$

But $x^2 \neq -4$ since x is real. $\quad$ $x=\pm 1$

sub into (2a)

when $x=1$ $\quad$ $x=-1$

$y=2$ $\quad$ $y=-2$

$\therefore$ $1+2i$ & $-1-2i$ are the square roots of $-3+4i$.

ie $\pm(1+2i)$ are the square roots of $-3+4i$

ii) $x^2 - (4-2i)x + (6-8i) = 0$

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$x = \frac{4-2i \pm \sqrt{(-(4-2i))^2 - 4(1)(6-8i)}}{2(1)}$

$x = \frac{4-2i \pm \sqrt{16-16i-4-24+32i}}{2}$

$x = \frac{4-2i \pm \sqrt{-12+16i}}{2}$

$x = \frac{4-2i \pm 2\sqrt{-3+4i}}{2}$

$x = 2-i \pm \sqrt{-3+4i}$

from(i)

$x = 2-i \pm (1+2i)$

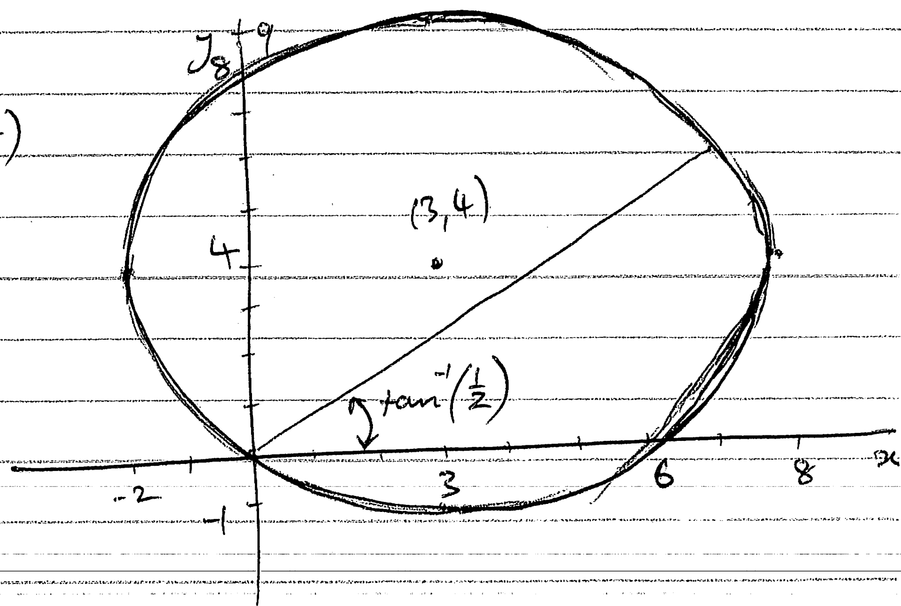
$x = 2-i+1+2i$ or $x = 2-i-1-2i$

$x = 3+i$ $\qquad$ $x = 1-3i$

$\therefore$ the roots are $3+i$ & $1-3i$.

d) i) $|z-(3+4i)| = 5$

is a circle centre $(3,4)$ radius 5.



ii) $\theta = \tan^{-1}\left(\frac{1}{2}\right)$

$\tan\theta = \frac{1}{2}$

$m = \frac{1}{2}$

$(x-3)^2 + (y-4)^2 = 25$ —— 2

$y = \frac{x}{2}$ —— 1

sub 1 into 2

$(x-3)^2 + \left(\frac{x}{2} - 4\right)^2 = 25$

$x^2 - 6x + 9 + \frac{x^2}{4} - 4x + 16 = 25$

$4x^2 - 24x + 36 + x^2 - 16x + 64 = 100$

$5x^2 - 40x + \cancel{100} = \cancel{100}$

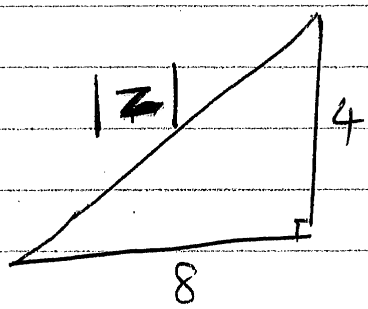
$5x(x-8) = 0$

$x = 0 \qquad x = 8$

sub into 1

$y = 0 \qquad y = 4.$

The line $y = \frac{x}{2}$ intersects the circle $(x-3)^2 + (y-4)^2 = 25$ at $(0,0)$ and $(8,4)$.



$|z| = \sqrt{8^2 + 4^2}$

$= \sqrt{64 + 16}$

$= \sqrt{80}$

$= 4\sqrt{5}$

QUESTION 3.

a) Let the polynomial be $x^3 - S_1x^2 + S_2x - S_3 = 0$ (A).

now $S_1 = \alpha+\beta+\gamma = 7$.

$S_3 = 5$

$S_1^2 = (\alpha+\beta+\gamma)^2 = \alpha^2+\beta^2+\gamma^2+2(\alpha\beta+\alpha\gamma+\beta\gamma)$

ie. $S_1^2 = \alpha^2+\beta^2+\gamma^2+2S_2$.

$49 = 29 + 2S_2$

$\therefore S_2 = 10$.

$\therefore$ (A) becomes $\boxed{x^3 - 7x^2 + 10x - 5 = 0}$

b) (i) $P(x) = x^4 + Ax^2 + B$ where A and B are positive

Clearly $P(x) \geq B \therefore P(x) > 0$ (x^4 and Ax^2 are ~~pos~~ non-negative)

$\therefore P(x) \neq 0$. $\therefore$ no real zeros.

NOTE * Those who treated $P(x)$ as a quadratic struggled to gain marks.

* Other approaches, commonly used, involved calculus to find that B is the minimum value.

(ii) Since the co-efficients are real, by the conjugate root theorem, the four roots are; $\pm ib$ & $\pm id$.

$$\therefore x^4 + Ax^2 + B \equiv (x+ib)(x-ib)(x+id)(x-id)$$
$$\equiv (x^2+b^2)(x^2+d^2)$$
$$\equiv x^4 + (b^2+d^2)x^2 + b^2d^2.$$

equating $b^2+d^2 = A$.

$b^2d^2 = B$

now $b^4+d^4 = (b^2+d^2)^2 - 2b^2d^2$

$$\therefore \boxed{b^4+d^4 = A^2 - 2B.}$$

NOTE. There were several other ways of doing this equation.

c (i) $f(x) = x^3 - 3ax + b$

$f'(x) = 3x^2 - 3a.$

For turning pts $f'(x) = 0$

ie $3(x^2-a) = 0$

$x^2 = a$

$x = \pm\sqrt{a}.$

$\therefore$ turning pts exist if x is real

ie. $a > 0$

(NB. to argue that x^2 is positive therefore $a > 0$ is not answering the question as asked)

$f(\sqrt{a}) = a\sqrt{a} - 3a\sqrt{a} + b$

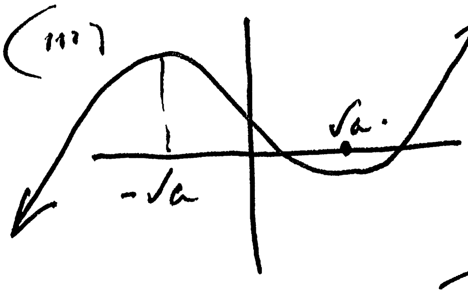
$= b - 2a\sqrt{a}.$

& $f(-\sqrt{a}) = -a\sqrt{a} + 3a\sqrt{a} + b$

$= b + 2a\sqrt{a}$

$\therefore$ turning points at $(\sqrt{a}, b - 2a\sqrt{a})$ and $(-\sqrt{a}, b + 2a\sqrt{a})$

(iii)



For three distinct real zeros $f(\sqrt{a})$ and $f(-\sqrt{a})$ need to be opposite in sign.

ie. $f(\sqrt{a}) \times f(-\sqrt{a}) < 0$.

$(b - 2a\sqrt{a})(b + 2a\sqrt{a}) < 0$.

$b^2 - 4a^3 < 0$

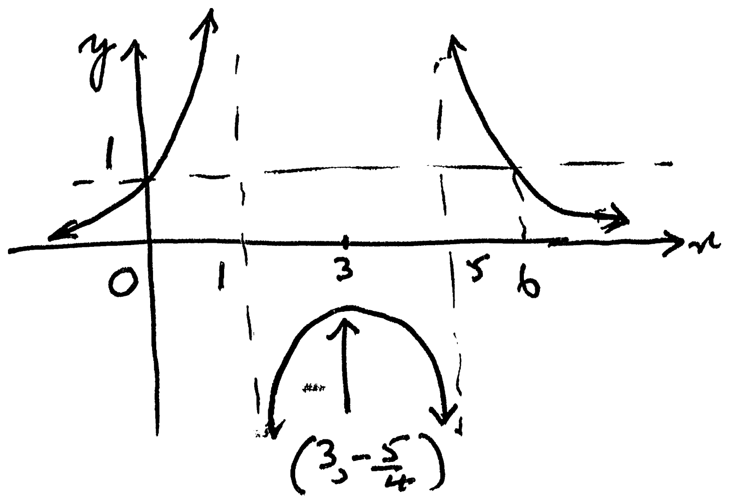
$\therefore \boxed{b^2 < 4a^3}$

QUESTION 4.

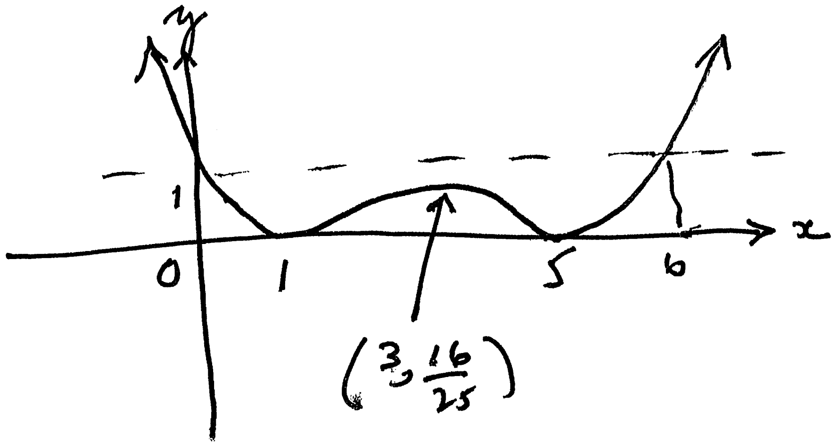
NB

Answers required zeros, turning pts, intercepts and asymptotes

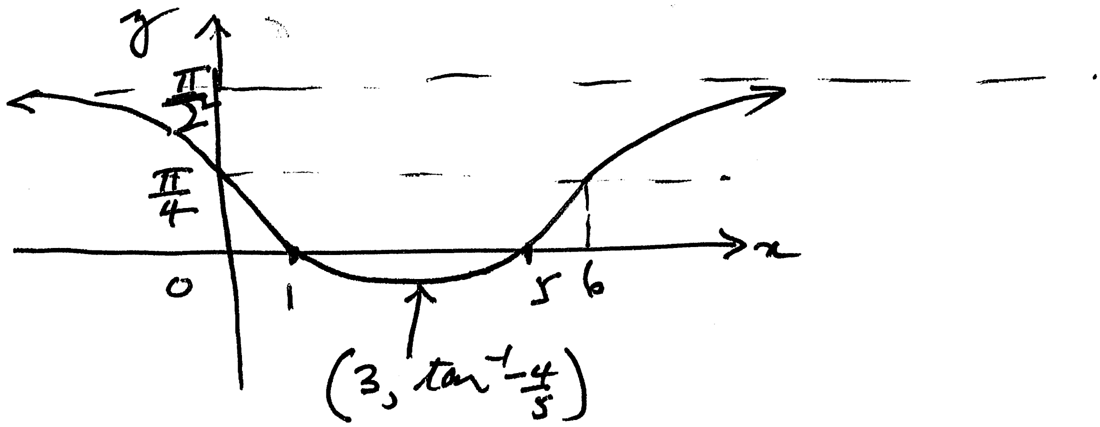
(i)



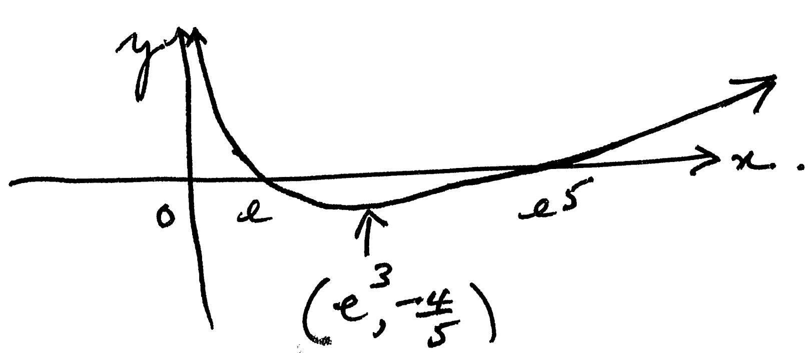
(ii)



(iii)



(iv)



(b) (i) now $f(x) = O(x) + E(x)$. — ①

& $f(-x) = O(-x) + E(-x)$. $\left(\begin{array}{l} \underline{\text{NB.}} \\ O(-x) = -O(x) \\ \& \ E(-x) = E(x) \end{array} \right)$.

ie $f(-x) = -O(x) + E(x)$. — ②

① − ②

$f(x) - f(-x) = 2\,O(x)$

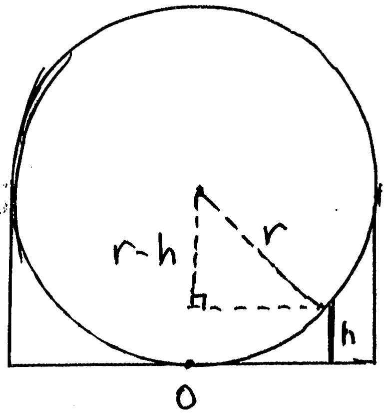
$$\therefore \boxed{O(x) = \frac{f(x) - f(-x)}{2}}.$$

(ii) If $f(x) = e^{x}$

$$O(x) = \boxed{\frac{e^{x} - e^{-x}}{2}} \quad \text{OR.} \quad \frac{e^{2x} - 1}{2e^{x}}$$

QUESTION 5

(a)
(i)



By Pythagoras' Theorem

$(r-h)^2 + x^2 = r^2$

ie $h^2 - 2rh + x^2 = 0$

As a quadratic in h

$\Rightarrow h = \dfrac{2r \pm \sqrt{4r^2 - 4x^2}}{2}$

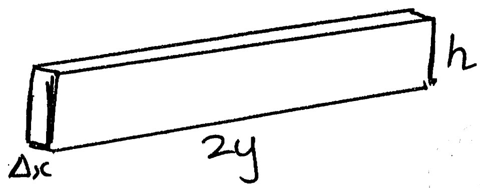
$h = r \pm \sqrt{r^2 - x^2}$

Since $h \leq r$

only $h = r - \sqrt{r^2 - x^2}$

applies

(ii) ΔV = Vol. of slice



$\Delta V = A\,\Delta x$

$\Delta V = 2yh\,\Delta x$

$= 2\sqrt{(r^2 - x^2)}\left[r - \sqrt{r^2 - x^2}\right]\Delta x$

ie $\Delta V = \left[2r\sqrt{r^2 - x^2} - 2(r^2 - x^2)\right]\Delta x$

(iii) Vol. of half-clamp V

$V = 2\int_0^r \left(2r\sqrt{r^2 - x^2} - 2r^2 + 2x^2\right)dx$

$= 2\left\{2r \cdot \frac{1}{4}\pi r^2 - \left[2r^2 x\right]_0^r + \left[\frac{2x^3}{3}\right]_0^r\right\}$

$= 2\left\{\frac{\pi}{2} r^3 - 2r^3 + \frac{2r^3}{3}\right\}$

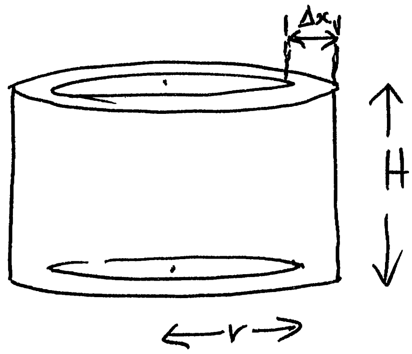
$V = \left\{\left(\frac{\pi}{2} - \frac{4}{3}\right) r^3\right\} \times 2$ units^3

$V = \pi r^3 - \frac{8}{3} r^3$

$\therefore$ Vol. of whole clamp (one piece)

(5)(b) inner radius of shell $= r$

outer radius of shell $= r + \Delta r$



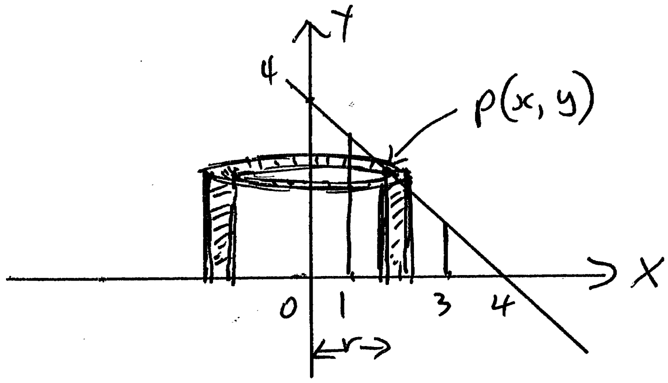
(i) $\Delta V =$ Vol. of shell

$$= \left[\pi(r+\Delta r)^2 - \pi r^2\right] H$$

$$= \pi\left[r^2 + 2r\Delta r + (\Delta r)^2 - r^2\right] H$$

$\Delta V = 2\pi r H \Delta r$ since $(\Delta r)^2$ is sufficiently small enough to be neglected.

(ii)



inner radius $= x$

height $= y$

$$\Delta V = 2\pi r H \Delta r$$

$\Delta V = 2\pi x y \Delta x$ where $x = r$, $y = H$

$$\therefore V = 2\pi \int_1^3 xy\,dx$$

$$= 2\pi \int_1^3 x(4-x)\,dx$$

$$= 2\pi \left[2x^2 - \frac{x^3}{3}\right]_1^3$$

$$= 2\pi \left[(18-9) - \left(2 - \tfrac{1}{3}\right)\right]$$

$$= 2\pi \times \frac{22}{3} = \frac{44\pi}{3} \text{ units}^3$$

Question 6

(a) let $S'(-2,0)$ $S(2,0)$ $T(x,y)$

(i)

$$TS' + TS = 6$$

$$\sqrt{(x+2)^2+y^2} + \sqrt{(x-2)^2+y^2} = 6$$

Squaring, rearranging, squaring

$$\Rightarrow \quad \frac{x^2}{9} + \frac{y^2}{5} = 1$$

Auxiliary circle of radius 3

(ii) centre $(0,0)$ is

$$x^2 + y^2 = 9$$

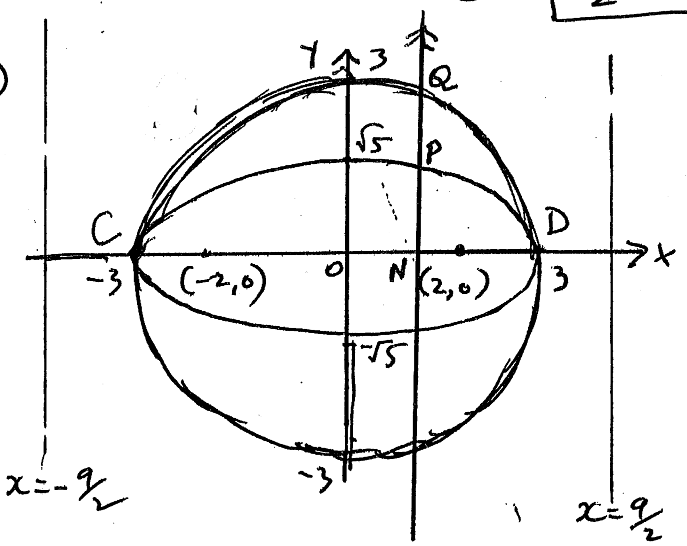
(iii) $b^2 = a^2(1-e^2)$

$$5 = 9(1-e^2) \Rightarrow \boxed{e = \frac{2}{3}}$$

Foci $(\pm ae, 0)$ ie $(\pm 2, 0)$

Directrices $x = \pm \frac{a}{e} \Rightarrow \boxed{\pm\frac{9}{2} = x}$

(iv)



(v) $N(3\cos\theta, 0)$

$P(3\cos\theta, \sqrt{5}\sin\theta)$

$Q(3\cos\theta, 3\sin\theta)$

(vi) Tangent at P is

$$\frac{x\cos\theta}{3} + \frac{y\sqrt{5}\sin\theta}{5\sin\theta} = 1 \quad (A)$$

Tangent at Q

grad. is $2x + 2yy' = 0$

$$y' = -\frac{x}{y}$$

at Q grad is $-\frac{3\cos\theta}{3\sin\theta} = -\cot\theta$

Eq^n is $y - 3\sin\theta = -\cot\theta(x - 3\cos\theta)$

(vii) $\Rightarrow y\sin\theta + x\cos\theta = 3 \quad (B)$

(α) Solving (A) and (B)

$$5x\cos\theta + 3\sqrt{5}y\sin\theta = 15 \quad -(A)$$

$$x\cos\theta + y\sin\theta = 3 \quad -(B)$$

$5\times(B) \Rightarrow 5x\cos\theta + 5y\sin\theta = 15 \quad -(C)$

$(A)-(C) \Rightarrow (3\sqrt{5}-5)y\sin\theta = 0$

$3\sqrt{5}-5 \neq 0$, $\sin\theta \neq 0$ except at C, D

$\Rightarrow \boxed{y=0}$ $\boxed{x = \frac{3}{\cos\theta}}$ $\cos\theta \neq 0$

$\therefore \forall x$ point $R\left(\frac{3}{\cos\theta}, 0\right)$ lies on the y axis.

(β) $ON.OR = 3\cos\theta \cdot \frac{3}{\cos\theta} = 9$

$\therefore$ independent of P and Q.

7. (a) (i) Show that $z^5+1=(z+1)\left(z^4-z^3+z^2-z+1\right)$. [1]

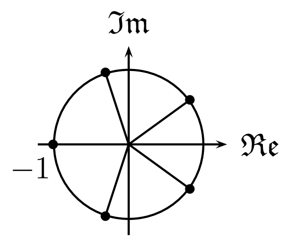
Solution:
$$\begin{aligned}\text{R.H.S.} &= z\left(z^4-z^3+z^2-z+1\right)+\left(z^4-z^3+z^2-z+1\right)\\ &= \left(z^5-z^4+z^3-z^2+z\right)+\left(z^4-z^3+z^2-z+1\right)\\ &= z^5+1\\ &= \text{L.H.S.}\end{aligned}$$

(ii) If z is a solution to $z^5+1=0$ where $z\neq -1$, prove that $1+z^2+z^4=z+z^3$. [1]

Solution:
$$\begin{aligned}(z+1)\left(z^4-z^3+z^2-z+1\right) &= 0 \quad \text{but } z\neq -1,\\ \therefore\ z^4-z^3+z^2-z+1 &= 0.\\ \text{Hence } 1+z^2+z^4 &= z+z^3.\end{aligned}$$

(iii) Hence show that $\cos\frac{\pi}{5}+\cos\frac{3\pi}{5}=\frac{1}{2}$. [3]

Solution:



From the diagram
$$\begin{aligned}\text{if } z^5 &= -1\\ z &= \text{cis} \pm \tfrac{\pi}{5},\ \text{cis} \pm \tfrac{3\pi}{5},\ -1\end{aligned}$$

Method 1:
$$\begin{aligned}\text{We take } z &= \text{cis}\,\tfrac{\pi}{5}.\\ 1+z^2+z^4 &= z+z^3 \text{ from (ii)},\\ \frac{1}{z^2}+1+z^2 &= \frac{1}{z}+z,\\ 2\cos\tfrac{2\pi}{5}+1 &= 2\cos\tfrac{\pi}{5},\\ 2\cos\tfrac{\pi}{5}-2\cos\tfrac{2\pi}{5} &= 1,\\ 2\cos\tfrac{\pi}{5}+2\cos\tfrac{3\pi}{5} &= 1,\\ \therefore\ \cos\tfrac{\pi}{5}+\cos\tfrac{3\pi}{5} &= \tfrac{1}{2}.\end{aligned}$$

Method 2: We consider the roots of $z^4-z^3+z^2-z+1=0$ from (ii), taken one-at-a-time,
$$\begin{aligned}\text{cis}\,\tfrac{\pi}{5}+\text{cis}\,\tfrac{-\pi}{5}+\text{cis}\,\tfrac{3\pi}{5}+\text{cis}\,\tfrac{-3\pi}{5} &= 1.\\ \text{But } z+\bar{z} &= 2\mathfrak{Re}(z),\\ \text{so cis}\,\tfrac{\pi}{5}+\text{cis}\,\tfrac{-\pi}{5} &= 2\cos\tfrac{\pi}{5} \text{ etc.}\\ \therefore\ 2\cos\tfrac{\pi}{5}+2\cos\tfrac{3\pi}{5} &= 1,\\ \text{and } \cos\tfrac{\pi}{5}+\cos\tfrac{3\pi}{5} &= \tfrac{1}{2}.\end{aligned}$$

(b) For integer values of k where $k = 0, 1, 2, \ldots$ define I_k as follows:

$$I_k = \int_0^{\frac{\pi}{2}} \cos^k x \, dx$$

(i) Express I_{k+2} in terms of k and I_k. $\boxed{2}$

Solution:

$$u = \cos^{k+1} \qquad v' = \cos x$$
$$u' = (k+1)(-\sin x)\cos^k x \qquad v = \sin x$$

$$\begin{aligned}
I_{k+2} &= \int_0^{\frac{\pi}{2}} \cos^{k+1} x . \cos x \, dx, \\
&= \left[\sin x \cos^{k+1} x\right]_0^{\frac{\pi}{2}} + (k+1)\int_0^{\frac{\pi}{2}} \sin^2 x . \cos^k x \, dx, \\
&= 0 + (k+1)\int_0^{\frac{\pi}{2}} \left(\cos^k x - \cos^{k+2} x\right) \, dx, \\
&= (k+1)I_k - (k+1)I_{k+2}, \\
(k+2)I_{k+2} &= (k+1)I_k, \\
I_{k+2} &= \left(\frac{k+1}{k+2}\right) I_k.
\end{aligned}$$

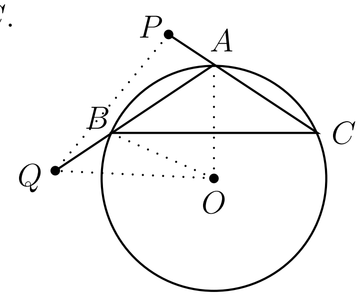
(ii) Hence find an expression for I_{2n}, where $n = 0, 1, 2, \ldots$ $\boxed{2}$

Solution:

$$\begin{aligned}
I_0 &= \int_0^{\frac{\pi}{2}} dx \quad = I_{2\times 0} \quad i.e.\ n = 0, \\
&= \frac{\pi}{2}. \\
I_2 &= I_{0+2} \quad = I_{2\times 1} \quad i.e.\ n = 1, \\
&= \frac{1}{2}.I_0, \\
&= \frac{1}{2}.\frac{\pi}{2}. \\
I_4 &= I_{2+2} \quad = I_{2\times 2} \quad i.e.\ n = 2, \\
&= \frac{3}{4}.I_2, \\
&= \frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}. \\
I_6 &= I_{4+2} \quad = I_{2\times 3} \quad i.e.\ n = 3, \\
&= \frac{5}{6}.I_4, \\
&= \frac{5}{6}.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}, \\
&= \frac{6}{6}.\frac{5}{6}.\frac{4}{4}.\frac{3}{4}.\frac{2}{2}.\frac{1}{2}.\frac{\pi}{2}, \\
&= \frac{1}{2^6}.\frac{6!}{(3!)^2}.\frac{\pi}{2}. \\
I_{2n} &= \frac{1}{2^{2n}}.\frac{(2n)!}{(n!)^2}.\frac{\pi}{2}.
\end{aligned}$$

(c) In $\triangle ABC$, in the diagram on the right, $AB = AC$.

Produce CA to P and AB to Q so that $AP = BQ$.



(i) Show that $\angle OAP = \angle OBQ$. $\boxed{3}$

Solution: In $\triangle$s AOC, AOB,

$$\begin{aligned} AC &= AB \text{ (data)}, \\ OA &= OB = OC \text{ (equal radii)}, \\ \therefore\ \triangle AOC &\equiv \triangle AOB \text{ (SSS)}, \\ \angle OAC &= \angle OBA \text{ (corresp. } \angle\text{s in congruent } \triangle\text{ s)}, \\ \angle OAP + \angle OAC &= 180^\circ\ (= \angle PAC), \\ \angle OBQ + \angle OBA &= 180^\circ\ (= \angle ABQ), \\ \therefore\ \angle OAP &= \angle OBQ. \end{aligned}$$

(ii) Prove that A, P, Q, and O, the centre of the circle, ABC are concyclic. $\boxed{3}$

Solution: In $\triangle$s OAP, OBQ,

$$\begin{aligned} AP &= BQ \text{ (data)}, \\ \angle OAP &= \angle OBQ \text{ (shown above)}, \\ OA &= OB \text{ (equal radii)}, \\ \therefore\ \triangle OAP &\equiv \triangle OBQ \text{ (SAS)}, \\ \angle APO &= \angle BQO \text{ (corresp. } \angle\text{s in congruent } \triangle\text{ s)}, \\ \angle BQO &= \angle AQO \text{ (common)}, \\ \therefore\ \angle APO &= \angle AQO \end{aligned}$$

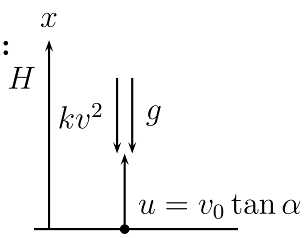
$\therefore$ A, P, Q, and O are concyclic
(equal angles subtended by the chord AO).

8. (a) A particle is projected **vertically** upwards in a resisting medium where the resistance varies as the square of the velovity and k is the constant of variation. If the velocity of projection is $v_0 \tan\alpha$,

(i) Show that the maximum height, H, reached is given by: $\boxed{3}$

$$H = \frac{1}{2k}\ln\left(\frac{g + kv_0^2\tan^2\alpha}{g}\right)$$

Solution:

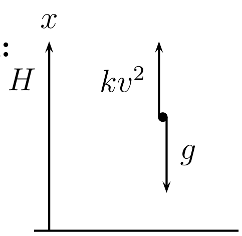


$$\begin{aligned}
\ddot{x} &= -kv^2 - g \\
v\frac{dv}{dx} &= -\left(kv^2 + g\right) \\
\frac{dv}{dx} &= -\frac{kv^2 + g}{v} \\
\int_0^H dx &= -\frac{1}{2k}\int_u^v \frac{2kv\,dv}{kv^2 + g} \\
0 - H &= -\frac{1}{2k}\left[\ln\left(kv^2 + g\right)\right]_u^0
\end{aligned}$$

Now, replacing u with $v_0\tan\alpha$, $H = \dfrac{1}{2k}\ln\left(\dfrac{g + kv_0^2\tan^2\alpha}{g}\right)$.

(ii) Show that the particle returns to the point of projection with velocity $v_0\sin\alpha$ given that v_0 is the terminal velocity. $\boxed{4}$

Solution:



$$\begin{aligned}
\ddot{x} &= kv^2 - g, \\
v\frac{dv}{dx} &= kv^2 - g.
\end{aligned}$$

At terminal velocity, $\ddot{x} = 0$,

$$\therefore\ kv_0^2 = g,$$

$$i.e.\ v_0^2 = \frac{g}{k} \quad \text{or} \quad v_0 = \sqrt{\frac{g}{k}}.$$

$$\begin{aligned}
\int_H^0 dx &= -\frac{1}{2k}\int_0^{-v} \frac{2kv\,dv}{kv^2 - g}, \\
0 - H &= \frac{1}{2k}\left[\ln\left(kv^2 - g\right)\right]_0^{-v}, \\
-H &= \frac{1}{2k}\ln\left(\frac{kv^2 - g}{-g}\right),
\end{aligned}$$

Now, equating Hs,

$$\frac{1}{2k}\ln\left(\frac{g + kv_0^2\tan^2\alpha}{g}\right) = \frac{1}{2k}\ln\left(\frac{g}{g - kv^2}\right),$$

$$\text{So,}\quad \frac{g + kv_0^2\tan^2\alpha}{g} = \frac{g}{g - kv^2},$$

$$
\begin{aligned}
1+\frac{kv_0^2}{g}.\tan^2\alpha &= \frac{1}{1-\frac{k}{g}v^2},\\
1+\tan^2\alpha &= \frac{1}{1-\frac{v^2}{v_0^2}},\\
\frac{\sec^2\alpha}{v_0^2} &= \frac{1}{v_0^2-v^2},\\
v_0^2-v^2 &= v_0^2\cos^2\alpha,\\
&= v_0^2-v_0^2\sin^2\alpha,\\
v^2 &= v_0^2\sin^2\alpha,\\
v &= v_0\sin\alpha.
\end{aligned}
$$

(iii) Show that the time of ascent is $\dfrac{v_0\alpha}{g}$ $\boxed{3}$

Solution: From part (i),

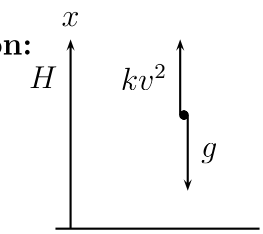
$$
\begin{aligned}
\ddot{x} &= -(g+kv^2),\\
\frac{dv}{dt} &= -(g+kv^2).\\
\int_0^T dt &= -\int_u^0 \frac{dv}{g+kv^2},\\
&= \frac{1}{k}\int_0^{v_0\tan\alpha}\frac{dv}{v_0^2+v^2},\ \text{using } \frac{g}{k}=v_0^2 \text{ from (ii)},\\
T &= \frac{1}{k}.\frac{1}{v_0}\left[\tan^{-1}\frac{v}{v_0}\right]_0^{v_0\tan\alpha},\\
&= \frac{1}{kv_0}\tan^{-1}\tan\alpha,
\end{aligned}
$$

$\therefore$ time of ascent,

$$
\begin{aligned}
T &= \frac{\alpha}{kv_0},\quad \text{but } k=\frac{g}{v_0^2},\\
&= \frac{\alpha}{v_0}.\frac{v_0^2}{g},\\
&= \frac{v_0\alpha}{g}.
\end{aligned}
$$

(iv) Show that the time of descent is $\dfrac{v_0}{g}\ln(\sec\alpha + \tan\alpha)$. $\boxed{2}$

Solution:



$$\begin{aligned}
\ddot{x} &= kv^2 - g,\\
\frac{dv}{dt} &= kv^2 - g.\\
\int_0^T dt &= \int_0^{-v_0\sin\alpha} \frac{dv}{kv^2 - g},\\
&= \frac{1}{k}\int_0^{-v_0\sin\alpha} \frac{dv}{v^2 - v_0^2}.
\end{aligned}$$

$$\begin{aligned}
\frac{1}{v^2 - v_0^2} &\equiv \frac{A}{v + v_0} + \frac{B}{v - v_0},\\
1 &\equiv A(v - v_0) + B(v + v_0).\\
\text{Put } v = v_0, &\qquad B = \frac{1}{2v_0},\\
\text{put } v = -v_0, &\qquad A = -\frac{1}{2v_0}.\\
\text{Then, } T &= \frac{1}{2kv_0}\int_0^{-v_0\sin\alpha}\left(\frac{1}{v - v_0} - \frac{1}{v + v_0}\right) dv,\\
&= \frac{1}{2kv_0}\Big[\ln(v - v_0) - \ln(v + v_0)\Big]_0^{-v_0\sin\alpha},\\
&= \frac{1}{2kv_0}\left\{\ln\left(\frac{-v_0\sin\alpha - v_0}{-v_0}\right) - \ln\left(\frac{v_0 - v_0\sin\alpha}{v_0}\right)\right\},\\
&= \frac{1}{2kv_0}\ln\left\{\left(\frac{\sin\alpha + 1}{1 - \sin\alpha}\right)\cdot\left(\frac{1 + \sin\alpha}{1 + \sin\alpha}\right)\right\},\\
&= \frac{1}{2kv_0}\ln\left\{\frac{(\sin\alpha + 1)^2}{\cos^2\alpha}\right\},\\
&= \frac{1}{kv_0}\ln\sqrt{(\sec\alpha + \tan\alpha)^2},\\
&= \frac{1}{kv_0}\ln(\sec\alpha + \tan\alpha).
\end{aligned}$$

(b) Prove by induction that, for all integers n where $n > 1$, that $\boxed{3}$

$$\frac{4^n}{n+1} < \frac{(2n)!}{(n!)^2}$$

Solution: Test for $n = 2$:

$$\text{L.H.S.} = \frac{4^2}{3}, \qquad \text{R.H.S.} = \frac{4!}{(2!)^2},$$
$$= 5\tfrac{1}{3}. \qquad\qquad = 6.$$

$\therefore$ True for $n = 2$.

Now, assume true for $n = k \geq 2$,

$$i.e. \quad \frac{4^k}{k+1} < \frac{(2k)!}{(k!)^2}$$

Then testfor $n = k+1$,

$$i.e. \quad \frac{4^{k+1}}{k+2} < \frac{(2k+2)!}{\big((k+1)!\big)^2}$$

$$\begin{aligned}
\text{L.H.S.} &= \frac{4.4^k}{k+1} \times \frac{k+1}{k+2}, \\
&< \frac{(2k)!}{(k!)^2} \times \frac{4(k+1)}{k+2} \quad \text{by the assumption,} \\
&< \frac{(2k+2)!}{\big((k+1)!\big)^2} \times \frac{(k+1)^2}{(2k+2)(2k+1)} \times \frac{4(k+1)}{(k+2)}, \\
&< \frac{(2k+2)!}{\big((k+1)!\big)^2} \times \frac{(2k^2+4k+2)}{(2k^2+5k+2)}, \\
&< \frac{(2k+2)!}{\big((k+1)!\big)^2} \times \left(1 - \frac{k}{2k^2+5k+2}\right), \\
&< \frac{(2k+2)!}{\big((k+1)!\big)^2} \quad \text{as} \quad \left(1 - \frac{k}{2k^2+5k+2}\right) < 1.
\end{aligned}$$

$$\begin{aligned}
\text{Alternatively,} \quad \text{R.H.S.} &= \frac{(2k)!}{(k!)^2} \times \frac{(2k+2)(2k+1)}{(k+1)^2}, \\
&> \frac{4^k}{k+1} \times \frac{2(2k+1)}{(k+1)} \quad \text{by the assumption,} \\
&> \frac{4^{k+1}}{k+2} \times \frac{(k+2)}{4(k+1)} \times \frac{2(2k+1)}{(k+1)}, \\
&> \frac{4^{k+1}}{k+2} \times \frac{(2k^2+5k+2)}{(2k^2+4k+2)}, \\
&> \frac{4^{k+1}}{k+2} \times \left(1 + \frac{k}{2k^2+4k+2}\right), \\
&> \frac{4^{k+1}}{k+2} \quad \text{as} \quad \left(1 + \frac{k}{2k^2+4k+2}\right) > 1.
\end{aligned}$$

$\therefore$ The statement is true for $n = k+1$ if true for $n = k$.
As true for $n = 2$, so true for $n = 3, 4, \ldots$ and so on for all natural numbers $n > 1$.